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NONLINEAR ANISOTROPIC TEMPERATURE DISTRIBUTION IN A WEDGE

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NOMENCLATURE

A, B, constants [dimensionless];
a, b, c, constants [cal/hcm°C];
C_0, C_1, constants of integration;
$F(T)$, function [dimensionless];
$G(T)$, function [°C];
$h(\eta)$, function [dimensionless];
k_{ij}, conductivity tensor [cal/hcm°C];
n, real number;
T, temperature [°C];
U, transformed temperature [°C];
x, y, coordinates [cm];
α, parameter [1/°C];
η, similarity variable [dimensionless];
ξ, function [°C].

INTRODUCTION

ANISOTROPIC materials are of widespread importance. Common examples are crystals, non crystalline substances such as sedimentary rocks and wood, and laminated materials such as plywood and transformer cores [1].

The analysis of conduction in anisotropic solids is given by Carslaw [2] and Özişik [3].

In this paper, the two dimensional temperature distribution in a wedge, having constant temperatures on its edges, is treated for the general nonlinear problem, the special nonlinear problem and the linear problem. In all three types the anisotropic, orthotropic, and isotropic cases are dealt with.

ANALYSIS

The partial differential equation of heat conduction for a two dimensional anisotropic material [4] is

$$\frac{\partial}{\partial x}\left[k_{11}\frac{\partial T}{\partial x}+k_{12}\frac{\partial T}{\partial y}\right]+\frac{\partial}{\partial y}\left[k_{21}\frac{\partial T}{\partial x}+k_{22}\frac{\partial T}{\partial y}\right]=0. \quad (1)$$

Where T = temperature, °C, and k_{ij} = conductivity tensor, cal/hcm°C. It is assumed that the k_{ij} are temperature dependent, so that $k_{ij} = k_{ij}(T)$.

To see if a similarity problem exists, we assume that $T = T(\eta)$, where $\eta = x/y^n$. Carrying out the indicated partial and ordinary differentiation, one is led to the nonlinear, second order, ordinary differential equation,

$$\frac{d}{d\eta}\left[[k_{11}(T)-\eta k_{12}(T)]\frac{dT}{d\eta}\right]+\left[1+\frac{\eta d}{d\eta}\right]\left[[-k_{21}(T)+\eta k_{22}(T)]\frac{dT}{d\eta}\right]=0 \quad (2)$$

if, and only if, $n = 1$, and $\eta = x/y$, where $0 \leqslant \eta \leqslant \infty$. Equation (2) is subject to the boundary conditions: (1) $T(\eta)_1) = T_1$, (2) $T(\eta_2) = T_2$. Thus a similarity problem exists in a wedge, and the permissible boundary conditions for equation (1), see Fig. 1, are: (1) $T(x, y = x/\eta_1) = T_1$, (2) $T(x = \eta_1 y, y) = T_1$, (3) $T(x, y = x/\eta_2) = T_2$, (4) $T(x = \eta_2 y, y) = T_2$.

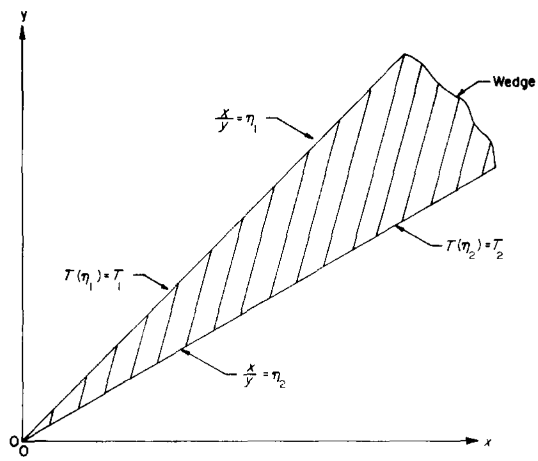

FIG. 1. Wedge details.

In the following, the treatment of equation (2) will be shown for three special categories.

I. *General nonlinear*

If $k_{ij} = k_{ij}(T)$, then equation (2) can be integrated in the anisotropic case, to give

$$\frac{dT}{d\eta}=C_0\Big/\left[k_{11}(T)-\eta[k_{12}(T)+k_{21}(T)]+\eta^2 k_{22}(T)\right]. \quad (3)$$

In the orthotropic case, we get equation (3) with $k_{12}(T) = 0 = k_{21}(T)$. In the isotropic case, we get the orthotropic case with $k_{11}(T) = k(T) = k_{22}(T)$. In equation (3), C_0 is a constant of integration, and the nonlinear, first order, ordinary differential equation can be solved by a Runge–Kutta method.

II. *Special nonlinear*

If $k_{ij} = k_{ij}F(T)$, then equation (3) can be separated, integrated, and written as

$$\int_{T_1}^{T} F(T)\,dT = G(T) - G(T_1)$$

$$= C_0 \int_{\eta_1}^{\eta} \frac{d\eta}{[k_{11} - \eta(k_{12} + k_{21}) + \eta^2 k_{22}]}$$

$$= C_1[h(\eta) - h(\eta_1)] \tag{4}$$

and using the boundary conditions, we can write,

$$U(T) = G(T) - G(T_1) - [G(T_2) - G(T_1)] \times [h(\eta) - h(\eta_1)]/[h(\eta_2) - h(\eta_1)] = 0 \tag{5}$$

where

$$h(\eta) = \arctan(A\eta + B). \tag{6}$$

For the anisotropic case, $A = 2a/\sqrt{(4ac - b^2)}$, $B = b/\sqrt{(4ac - b^2)}$, and for the orthotropic case, $A = \sqrt{a/c}$, $B = 0$, and for the isotropic case, $A = 1$, $B = 0$. For all three cases $a = k_{22}$, $b = -(k_{12} + k_{21})$ and $c = k_{11}$. Additionally, for all three cases, $T = U^{-1}[U(T)]$, where U^{-1} is the inverse of U, and must exist. If $F(T)$ of equation (4) is polynomial in form, and if the zero order term of $F(T)$ is not zero, the inverse U^{-1} always exists [5]. In applications $F(T)$ is usually a constant, or linear or quadratic in T.

III. *Linear*

If k_{ij} = constant, then equation (5) can be written as

$$T(\eta) = T_1 + (T_2 - T_1)[h(\eta) - h(\eta_1)]/[h(\eta_2) - h(\eta_1)] \tag{7}$$

where $h(\eta)$ is given in equation (6), and A and B for the anisotropic, orthotropic and isotropic case are the same as for the special nonlinear case just discussed. The solution to the linear isotropic case is well known, and is discussed by Churchill [6].

Example

Given equation (1), and the boundary conditions, (1) $T(\eta_1) = 100$, (2) $T(\eta_2) = 0$, where $\eta_1 = 0{\cdot}1$ and $\eta_2 = 10$, find the temperature distribution in the wedge ($0{\cdot}1 \leqslant \eta \leqslant 10$), for anisotropic, orthotropic and isotropic cases, when for the anisotropic case $k_{11} = k_{12} = k_{21} = 1 + 0{\cdot}001T$, and $k_{22} = 2 + 0{\cdot}002T$, for the orthotropic case, $k_{11} = 1 + 0{\cdot}001T$, $k_{22} = 2 + 0{\cdot}002T$, and $k_{12} = 0 = k_{21}$, and for the isotropic case, $k_{11} = k_{22} = 1 + 0{\cdot}001T$, and $k_{12} = 0 = k_{21}$.

The solution in all three cases, is

$$T = \frac{1}{\alpha}[\sqrt{(1 + 2\alpha\xi)} - 1] \tag{8}$$

where

$$\xi = 105\{1 - [h(\eta) - h(\eta_1)]/[h(\eta_2) - h(\eta_1)]\} \tag{9}$$

and where $\alpha = 0{\cdot}001$, and $h(\eta)$ is given by equation (6), and the values of A and B are those for the anisotropic, orthotropic, and isotropic case as discussed. The solutions are shown graphically in Fig. 2.

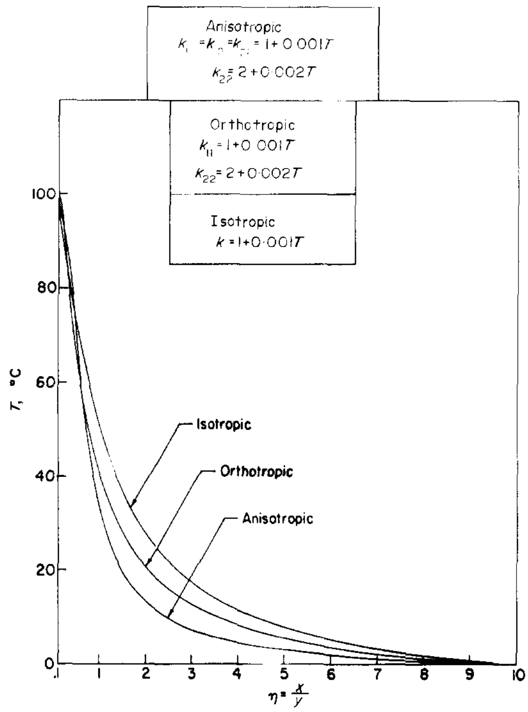

FIG. 2. Temperature distribution in a wedge.

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